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Stability of simultaneously block triangularisable switched systems with partial state reset

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ABSTRACT

We study the stability of a certain class of switched systems where discontinuous jumps (resets) on some of the state components are allowed, at the switching instants. It is known that, if all components of the state are available for reset, the system can be stabilisable by an adequate choice of resets. However, this question may have negative answer if there are forbidden state components for reset. We give a sufficient condition for the stabilisability of a switched system, under arbitrary switching, by partial state reset in terms of a block simultaneous triangularisability condition. Based on this sufficient condition, we show that the particular class of systems with partially commuting stable system matrices is stabilisable by partial state reset. We also provide an algorithm that allows testing whether a switched system belongs to this particular class of systems.

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1. Introduction

Switched systems may arise either as a direct result of the mathematical modelling of a phenomenon or as the result of controller design techniques that use switching schemes (see, for instance, Hespanha & Morse, 2002; Hespanha, Santesso, & Stewart, 2007; Liberzon & Morse, 1999). Such schemes rely on a bank of controllers supervised by a switching signal that indicates which controller should be operational at each time instant. Switching among the controllers in the bank results in a switching closed-loop system, and stability is not guaranteed even when each controller can stabilise the plant on its own.

In this paper, we consider switched linear systems; these systems are defined by a family of linear time-invariant (LTI) state-space systems (the bank of the switched system) together with a switching law, that determines which linear system within the family is active at each time instant.

When dealing with switched systems, the most common approach is to assume that, at each switching instant, the state evolves continuously (see, for instance, Liberzon, 2003; Liberzon & Morse, 1999). However, in some cases, it is natural and profitable to allow discontinuous state jumps (state resets) at the switching instants. Note that state resets, also known as *impulses*, are present not only in switched systems, but also in many other dynamical systems where the process may experience abrupt

state changes, as, for instance, in the case of some drug administration procedures (Lakshmikantham, Bainov, & Simeonov, 1989). In the majority of these situations, the reset is either determined by the trajectory of the system (Beker, Hollot, Chait, & Han, 2004), or it is time dependent, i.e. it occurs at some pre-specified time instants (regardless of the values of the state trajectory) (Hetel, Daafouz, Tarbouriech, & Prieur, 2013). In the case of switched systems, the state reset is usually associated with the switching signal. More precisely, the reset of the state takes place at each switching instant, and is determined by which systems are active before and after the commutation (Hespanha & Morse, 2002; Hespanha et al., 2007; Yuan & Wu, 2015a). In Hespanha and Morse (2002), the authors propose to make a reset in the state of a multi-controller in order to stabilise a given plant, this corresponds to considering that at each switching instant only some components of the state of the closed-loop system are available for reset, as the state of the process under control remains unchanged. Similar controller architecture may be found in Yuan and Wu (2015a, 2015b).

A switched system for which only some of the state components may jump during switching is called a *switched system with partial state reset* (Brás, Carapito, & Rocha, 2013).

One major issue when dealing with switched systems (with or without reset) is stability. Indeed, this property may not hold for the overall switched system even

when each individual LTI system in the bank is stable. A stability analysis of switched systems with reset may be carried out from two standpoints. One is to consider pre-specified (partial) reset and to study the system stability, the other is to investigate how to stabilise the system, when possible, by finding a suitable (partial) reset (stabilisation by reset). In this paper, the second standpoint is adopted: we consider that the switching signal is arbitrary and study the stabilisation by partial reset, in line with what is done in Brás et al. (2013). Note that, in many other approaches, see, for instance, Zhao, Yin, Li, and Niu (2015) and Zhao, Zhang, Shi, and Liu (2012), where the switching is not arbitrary, the stabilisation is made by picking suitable switching signals.

When dealing with the stability of a switched system (without reset), under arbitrary switching, it is frequent to use the common Lyapunov quadratic function approach. In fact, if, at each switching instant, the state evolves continuously, the existence of a common quadratic Lyapunov function (CQLF) is sufficient to ensure the stability of the switched system. Several survey papers on this matter are available (see, for example, Liberzon & Morse, 1999; Lin & Antsaklis, 2009; Shorten, Wirth, Mason, Wulff, & King, 2007). The books, Liberzon (2003) and Sun (2006), are also good references on the subject.

In Brás et al. (2013), the case was considered where the switching bank consists of stable LTI state-space systems with block triangular system matrices. In that paper, it was proved that in this case a system is stabilisable by partial reset if some relevant diagonal blocks of the system matrices have CQLF; moreover, a choice of suitable stabilising resets was provided.

In this paper, we are interested in investigating how to ensure stability by means of partial reset for switched systems associated to banks with simultaneously block triangularisable stable matrices. Our idea is to take advantage of the results from Brás et al. (2013). One might think that this latter problem can be easily reduced to the former one, as it would be enough to find, if possible, a simultaneously block triangularising similarity transformation for all the system matrices in the bank; but it turns out that some similarity transformations may destroy the property of stabilisation by partial reset. However, here we show that if the system bank is simultaneously block triangularisable by a similarity transformation with a (1,1) invertible block and, moreover, the correspondent (1,1) blocks in the bank system matrices have CQLF, then the system is stabilisable by a partial reset. Additionally, a way to construct suitable resets in order to ensure stability is given. Based on this result, we identify a class of switched systems, that we call partially commuting systems, which are stabilisable by a partial reset of a certain order (the order of the reset is the number of components of the state

that are allowed to jump). Note that, as shall become clear, the class of partially commuting systems is wider than the one of the systems with simultaneously triangularisable matrices.

The paper is organised as follows. Preliminaries are provided in Section 2. The definition of a set of simultaneously z -block triangularisable matrices and a sufficient condition for the stabilisation by partial reset of a switched system associated to a simultaneously block triangularisable bank are given in Section 3. In Section 4, based on the characterisation of partially commuting matrices, a class of simultaneously z -block triangularisable systems is identified for which stability property is guaranteed by partial reset. Finally, a numerical example is given followed by an algorithm that determines the smallest order of a stabilising partial reset that can be constructed based on the simultaneous block triangularisation of a given set of partially commuting matrices. Conclusions are given in Section 5. In Appendix A, a detailed exposition about partially commuting matrices is given in order to clarify some of the algebraic contents of the paper.

2. Preliminaries

In this section, we formalise the definition of switched system (with or without reset), introduce the stability property under consideration and summarise some relevant results on stability and stabilisation by reset.

Let \mathcal{P} a finite index set, $\Sigma_{\mathcal{P}} = \{\Sigma_p : p \in \mathcal{P}\}$ a family of LTI state-space systems. Since we focus on the study of internal stability, we consider systems without inputs and disregard the output equation. Let $\dot{x}(t) = A_p x(t)$, where $A_p \in \mathbb{R}^{n \times n}$ be the state-space representation of Σ_p , for $p \in \mathcal{P}$. Additionally, define a *switching law* or a *switching signal* as a piecewise constant function of time $\sigma : [t_0, +\infty[\rightarrow \mathcal{P}$, such that, $\sigma(t) = i_{k-1}$, for $t \in [t_{k-1}, t_k[$, $k \in \mathbb{N}$; the time instants t_k , $k \in \mathbb{N}$, such that $t_0 < t_1 < \dots < t_{k-1} < t_k \dots$ are called *switching instants*. The set of all switching signals is represented by $\mathcal{S}_{\mathcal{P}}$. A triple $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ is said to be a *switched system with switching bank* $\Sigma_{\mathcal{P}}$, in the sense that each switching signal $\sigma(\cdot) \in \mathcal{S}_{\mathcal{P}}$ produces the linear time-varying system Σ_{σ} defined by

$$\dot{x}(t) = A_{\sigma(t)} x(t) \quad (1)$$

for all $t \geq t_0$, where $x(t) \in \mathbb{R}^n$ is the state.

This definition implies that, as is commonly assumed in most of the literature, there are no state jumps at the switching instants. Next, we present the definition of switched system with state reset, that allows a state jump at each of the switching instants; this is uniquely

determined by the systems Σ_p that are active before and after the switching. The state discontinuities are associated with a family of resets:

$$\mathcal{R} = \{(q, p, R_{(q,p)}) : (q, p) \in \mathcal{P} \times \mathcal{P}, q \neq p\},$$

where $R_{(q,p)}$ are non-zero real matrices of order n that act on the state of the system at the switching instants. These matrices are called *reset matrices*.

Definition 2.1: A quadruple $\mathbb{S}_R = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R})$ is said to be a *switched system with state reset* if each switching signal $\sigma(\cdot) \in \mathcal{S}_{\mathcal{P}}$ produces a linear time-varying system defined as in (1), such that at each switching time instant t_k , for all $k \in \mathbb{N}$, with $t_0 < t_1 < \dots < t_{k-1} < t_k < \dots$,

$$x(t_k) = R_{(i_{k-1}, i_k)} x(t_k^-), \quad \text{if } \sigma(t) = i_{k-1} \text{ for } t \in [t_{k-1}, t_k[, \quad (2)$$

where $x(t_k^-) := \lim_{t \rightarrow t_k^-} x(t)$ and $R_{(i_{k-1}, i_k)}$ are reset matrices taken according with the family of resets \mathcal{R} .

Clearly, according to Definition 2.1, if, for all $(q, p) \in \mathcal{P} \times \mathcal{P}$, $R_{(q,p)} = I_n$, where I_n denotes the identity matrix of order n , the correspondent switched system is a system without state reset, i.e. a classical switched system where the state evolves continuously. Note that for these systems (without reset) we use the notation $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$, as already mentioned.

Next, we present the definition of stability of a switched system (with or without state reset), under arbitrary switching. This notion is the usual global uniform exponential stability (Liberzon, 2003, p. 22), which for linear systems with arbitrary switching is equivalent to the property of global uniform asymptotic stability (see Liberzon, 2003, p. 22). In this paper for simplicity reasons, we use the term ‘stable’ to refer to both equivalent properties.

Definition 2.2: A switched system defined by (1) and (2) is *stable* if there exist $\gamma, \lambda \in \mathbb{R}^+$ such that, for every switching signal σ , for every $t_0 \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^n$, the correspondent state trajectory $x(t)$, with $x(t_0) = x_0$, satisfies $\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)} \|x_0\|$, for $t \geq t_0$.

For a switched system to be stable, it is necessary to have a bank of stable invariant systems. So, in the following, we exclusively consider switched systems that have a switching bank $\Sigma_{\mathcal{P}}$ formed by stable systems.

On the other hand, a well-known condition that guarantees the stability of a switched system \mathbb{S} is based on the notion of CQLF. A function $v(x) = x^T P x$, where P is a square symmetric positive definite matrix, is said to be a CQLF for a switched system \mathbb{S} if $A_p^T P + P A_p < 0$, for all $p \in \mathcal{P}$. With some abuse of language, we shall call the matrix P a CQLF for $\Sigma_{\mathcal{P}}$ and $A_p, p \in \mathcal{P}$. In fact, if at each switching instant, the state evolves continuously,

the existence of a common Lyapunov quadratic function is sufficient to ensure stability of the switched system, under arbitrary switching (Khalil, 1992; Liberzon & Morse, 1999).

Theorem 2.1: *If there exists a CQLF for the switched system \mathbb{S} , then \mathbb{S} is stable.*

Among the results related with the existence of a CQLF for a switched system (see, for instance, Liberzon, Hespanha, & Morse, 1999; Narendra & Balakrishnan, 1994; Shorten & Narendra, 1998; Shorten & Narendra, 2002), one of the most general sufficient conditions that can be stated in simple matrix terms is the following.

Theorem 2.2: *Let $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ be a switched system for which the switching bank $\Sigma_{\mathcal{P}}$ is associated to a set of stable simultaneously triangularisable matrices. Then there exists a CQLF for the switched system \mathbb{S} .*

Recalling that commutative matrices are simultaneously triangularisable (Motzkin & Taussky, 1952), the next result is a trivial consequence of Theorem 2.2 (Narendra & Balakrishnan, 1994), which will be relevant in the sequel.

Corollary 2.1: *A set $\{A_p : p \in \mathcal{P}\}$ of stable and pairwise commuting matrices has a CQLF.*

In an attempt to relax the conditions of Theorem 2.2, some sufficient conditions were established in terms of block matrices where the structure is block triangular or simultaneously block triangularisable (see, for instance, Brás, Carapito, & Rocha, 2006; Cheng, Guo, & Huang, 2003). The following results are closely related to the approach used in the current paper.

Theorem 2.3: *Let $\{A_p : p \in \mathcal{P}\}$ be a set of stable matrices with block triangular structure, where all diagonal blocks $A_{ii}^p, i = 1, \dots, t$, are square of the same order, for all $p \in \mathcal{P}$. If there exists a CQLF for each set of blocks $\{A_{ii}^p : p \in \mathcal{P}\}$, for $i = 1, \dots, t$, then all matrices A_p share a CQLF.*

Notice that, it is easy to conclude that the condition of previous theorem becomes a necessary and sufficient condition for the existence of CQLF with block diagonal structure.

Corollary 2.2: *Let $\{A_p : p \in \mathcal{P}\}$ be a set of stable matrices. If the matrices A_p are simultaneously block triangularisable, by a complex similarity transformation T , in such way that $\{T^{-1} A_p T : p \in \mathcal{P}\}$ satisfies the conditions of Theorem 2.3, then all matrices A_p share a CQLF.*

Another result related to the existence of CQLF is the following (see Brás et al., 2013, for a proof).

Lemma 2.1: *Let $\{A_p : p \in \mathcal{P}\} \subset \mathbb{R}^{n \times n}$ be a set of stable matrices and $P \in \mathbb{R}^{n \times n}$ a symmetric and positive definite*

matrix. Then there exists a set of invertible real matrices $\{S_p : p \in \mathcal{P}\}$ such that $\bar{A}_p = S_p^{-1}A_pS_p$ share P as a CQLF.

Using the previous lemma, it is possible to establish the following result that ensures the stability of any switched system provided that a proper choice of resets is made and if all components of the state can be freely reset (Brás et al., 2013).

Theorem 2.4: Let $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ be a switched system and S_p be invertible real matrices such that $S_p^{-1}A_pS_p$ share a CQLF. If $\mathcal{R} = \{(q, p, S_pS_q^{-1}) : (q, p) \in \mathcal{P} \times \mathcal{P}, q \neq p\}$, then, $\mathbb{S}_{\mathcal{R}} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R})$ is a stable system.

This theorem shows that a switched system, with a bank formed by stable systems, is always stabilisable using invertible reset matrices, if all state variables are allowed to jump. However, as already mentioned, in some situations the reset of some of those components may be forbidden or inconvenient. This leads to the analysis of switched systems with state reset with a certain block structure, i.e. switched systems where the reset matrices in \mathcal{R} are of the form

$$R_{(q,p)} = \begin{bmatrix} I_{n-z} & 0 \\ R_{21}^{(q,p)} & R_{22}^{(q,p)} \end{bmatrix}, \quad (3)$$

where the matrices $R_{22}^{(q,p)}$ are real invertible matrices of order z . In opposition to the case in which all of the state components can be subject to reset (total reset), we say that this type of reset is a *partial reset of order z* . This type of state reset appears, for instance, in Hespanha and Morse (2002) and Yuan and Wu (2015a), as a result of a multi-controller technique where the state of the controller is subject to some jump at the switching instants but the same is not allowed for the state of the process. In fact, in the aforementioned contributions, the correspondent closed-loop systems are switched systems with partial reset of the form $\begin{bmatrix} I & 0 \\ 0 & \tilde{R}_{(q,p)} \end{bmatrix}$.

Contrary to what happens when all state components are free for reset, stability is not always achievable when only partial resets, as in (3), are allowed (see Example 1 in Brás et al., 2013).

Definition 2.3: A switched system $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ is said to be *stabilisable by state reset* if there exists a family of resets \mathcal{R} such that the switched system with state reset $\mathbb{S}_{\mathcal{R}} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R})$ is stable. In particular, if each reset matrix of \mathcal{R} is an invertible matrix of the form

$$\begin{bmatrix} I_{n-z} & 0 \\ R_{21}^{(q,p)} & R_{22}^{(q,p)} \end{bmatrix}, (q, p) \in \mathcal{P} \times \mathcal{P}, z = 1, 2, \dots, n, \quad (4)$$

the system \mathbb{S} is called stabilisable by partial state reset of order z . When $z = n$, the partial reset becomes a total one.

As we shall see in the next section, the stabilisation by partial resets is not preserved under arbitrary similarity transformations, contrary to what happens in the case of total reset. However, some classes of similarity transformations preserve that property. In the following, we explore this fact in order to give some sufficient conditions for stabilisation by partial reset.

3. Stabilisation by partial resetting and simultaneously z-block triangularisable switched systems

The following theorem gives a sufficient condition for the stabilisation by partial resets for switched systems with block triangular structure; it is an immediate consequence of Corollaries 1 and 2 from Brás et al. (2013).

Theorem 3.1: Let $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ be a block triangular switched system associated to a set of stable matrices $A_p = \begin{bmatrix} A_{p11} & A_{p12} \\ 0 & A_{p22} \end{bmatrix}$, with $A_{p11} \in \mathbb{R}^{(n-z) \times (n-z)}$, $p \in \mathcal{P}$. If $\{A_{p11} : p \in \mathcal{P}\}$ has a CQLF, then \mathbb{S} is stabilisable by partial reset with order z .

From this result, a natural question arises: does the conclusion of Theorem 3.1 still hold if the matrices A_p are not block triangular but rather simultaneously block triangularisable? This is not a trivial question since, as the next example shows, a similarity transformation does not necessarily preserve the property of stabilisation by partial reset.

Example 3.1: The switched system with switching bank constituted by the matrices $A_p = \begin{bmatrix} -I_2 & 0 \\ 0 & A_{p22} \end{bmatrix}$, for $p = 1, 2$, with $A_{122} = \begin{bmatrix} -0.05 & 2 \\ -1 & -0.05 \end{bmatrix}$ and $A_{222} = \begin{bmatrix} -0.05 & 1 \\ -2 & -0.05 \end{bmatrix}$ is stabilisable by partial resets of order 2 but the switched system with switching bank associated to the matrices $T^{-1}A_pT$, with $T = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}$, is not. Indeed, we have $T^{-1}A_pT = \begin{bmatrix} A_{p22} & 0 \\ 0 & -I_2 \end{bmatrix}$ and the switched system associated to the blocks $(1,1)$ is unstable (Santos, 2002). So, the switched system associated to the matrices $T^{-1}A_pT$ is not stabilisable by partial resets of order 2.

Consequently, if we want to successfully study the partial stabilisation of a switched system by applying a similarity transformation, we should identify a suitable class of transformations that preserve the partial stabilisation property; moreover, the transformed system should be easier to analyse, with respect to partial stabilisation. In the following, we shall see that such strategy can be followed with success for a certain type of simultaneous block triangularisation matrices, namely those that are real and have an invertible upper left block.

Definition 3.1: The matrices $A_p \in \mathbb{R}^{n \times n}$, $p \in \mathcal{P}$, are said to be *simultaneously z-block triangularisable* if there exists an invertible real matrix $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$, where $T_{11} \in \mathbb{R}^{(n-z) \times (n-z)}$ is invertible matrix, such that $T^{-1}A_p T = \begin{bmatrix} \tilde{A}_{p11} & \tilde{A}_{p12} \\ 0 & \tilde{A}_{p22} \end{bmatrix}$, where $\tilde{A}_{p11} \in \mathbb{R}^{(n-z) \times (n-z)}$.

A switched system with a switching bank associated to a set of simultaneously z-block triangularisable matrices is said to be a *simultaneously z-block triangularisable switched system*. Notice that, if T is the identity matrix, the matrices are block triangular.

Theorem 3.2: Let $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ be a switched system associated to stable matrices A_p , $p \in \mathcal{P}$. If \mathbb{S} is simultaneously z-block triangularisable by the matrix T and all blocks (1,1) of order $n - z$ of the matrices $T^{-1}A_p T$ have a CQLF, then \mathbb{S} is stabilisable by partial reset of order z .

Proof: Let $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ be a real invertible matrix, where T_{11} is also invertible, such that

$$\tilde{A}_p = T^{-1}A_p T = \begin{bmatrix} \tilde{A}_{p11} & \tilde{A}_{p12} \\ 0 & \tilde{A}_{p22} \end{bmatrix}$$

and the blocks \tilde{A}_{p11} have a CQLF. By Lemma 2.1, there exists a set of invertible real matrices W_p , $p \in \mathcal{P}$, such that $W_p^{-1}\tilde{A}_{p22}W_p$ have a CQLF.

Consider $\tilde{S}_p = \begin{bmatrix} I & -T_{11}^{-1}T_{12}W_p \\ 0 & W_p^{-1}\tilde{A}_{p22}W_p \end{bmatrix}$, then $\tilde{S}_p^{-1} = \begin{bmatrix} I & T_{11}^{-1}T_{12} \\ 0 & W_p^{-1} \end{bmatrix}$. Therefore,

$$\begin{aligned} \bar{A}_p &= \tilde{S}_p^{-1}\tilde{A}_p\tilde{S}_p \\ &= \begin{bmatrix} \tilde{A}_{p11} & -\tilde{A}_{p11}T_{11}^{-1}T_{12}W_p + \tilde{A}_{p12}W_p + T_{11}^{-1}T_{12}\tilde{A}_{p22}W_p \\ 0 & W_p^{-1}\tilde{A}_{p22}W_p \end{bmatrix} \end{aligned}$$

Since \tilde{A}_{p11} , $p \in \mathcal{P}$, have a CQLF as well as $W_p^{-1}\tilde{A}_{p22}W_p$, $p \in \mathcal{P}$, by Theorem 2.3, \bar{A}_p , $p \in \mathcal{P}$, have also a CQLF. Therefore, there exists S_p such that $S_p^{-1}A_p S_p$ share a CQLF; more concretely,

$$\begin{aligned} S_p &= T\tilde{S}_p \\ &= \begin{bmatrix} T_{11} & 0 \\ T_{12} & CW_p \end{bmatrix}, \end{aligned}$$

$C = T_{22} - T_{21}T_{11}^{-1}T_{12}$ is the Schur complement of the block T_{11} in T . Then, by Theorem 2.4, the system $\mathbb{S}_R =$

$(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R})$ is stable, for the partial resets:

$$\begin{aligned} R_{(q,p)} &= S_p S_q^{-1} \\ &= \begin{bmatrix} T_{11} & 0 \\ T_{12} & CW_p \end{bmatrix} \begin{bmatrix} T_{11}^{-1} & 0 \\ -W_q^{-1}C^{-1}T_{12}T_{11}^{-1} & W_p^{-1}C^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ (I - CW_p W_q^{-1}C^{-1}) & T_{12}T_{11}^{-1} & CW_p W_q^{-1}C^{-1} \end{bmatrix}, \end{aligned}$$

so \mathbb{S} is stabilisable by partial reset of order z . ■

Notice that, the previous proof is constructive. In fact, it gives a way to construct a set of invertible partial reset matrices that stabilises the system.

4. Partially commuting switched systems

In this section, we identify a class of systems that are simultaneously z-block triangularisable and for which the transformed (1,1) blocks share a CQLF. Note that, in order to have a z-block triangularisable switched system, the following conditions should be satisfied:

- (1) the matrices A_p , $p \in \mathcal{P} = \{1, \dots, N\}$, must have a common invariant subspace \mathcal{N} of dimension $n - z$;
- (2) if $T = [v_1 \dots v_{n-z} \dots v_n]$, where $\{v_1, \dots, v_{n-z}\}$ is a basis of \mathcal{N} and $\{v_1, \dots, v_n\}$ is a basis of \mathbb{R}^n , and hence $T^{-1}A_p T = \begin{bmatrix} \tilde{A}_{p11} & \tilde{A}_{p12} \\ 0 & \tilde{A}_{p22} \end{bmatrix}$, then $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ is a real matrix with T_{11} invertible.

If, additionally, the (1,1) blocks \tilde{A}_{p11} commute pairwise, by Corollary 2.1, they have a CQLF. So, using Theorem 3.2, we may conclude that the system is stabilisable by partial reset of order z . Bearing this in mind, we are interested in characterising the sets of matrices $\{A_p : p \in \mathcal{P}\}$ that share a common invariant subspace where they commute; this is equivalent to the existence of a common eigenvector. A characterisation of the sets of two matrices with a common eigenvector was made in Shemesh (1984) and generalised for a finite number of matrices in Jamiołkowski and Pastuszak (2015). Matrices that have a common eigenvector are said to be partially commuting matrices, (Alpin, Elsner, & Ikramov, 2000). See Appendix A for more details on this. Note that, simultaneous triangularisable matrices and pairwise commutative matrices are particular classes of partially commuting matrices. In the sequel, for the sake of simplicity, we consider systems where $N = 2$. For our purposes, the following characterisation of partially commuting matrices is the most suitable one (it is an immediate consequence of Theorem A.1 of Appendix A).

Theorem 4.1: Let $A_1, A_2 \in \mathbb{R}^{n \times n}$ and $L = \sum_{k,l=1}^{n-1} [A_1^k, A_2^l]^T [A_1^k, A_2^l]$. The matrices A_1 and A_2 are partially commuting matrices if and only if $\text{rank } L < n$.

The following corollary gives an easy test for the stabilisability by partial reset of order z of a partially commuting system for which $\text{rank } L = z$.

Corollary 4.1: Let $A_1, A_2 \in \mathbb{R}^{n \times n}$ be partially commuting stable matrices such that $\text{rank } L = z$ and let $\{v_1, \dots, v_{n-z}\}$ be a basis of $\ker L$. If $\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} = [v_1 \dots v_{n-z}]$, where $T_{11} \in \mathbb{R}^{(n-z) \times (n-z)}$ is invertible, then the switched system $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ associated to A_1 and A_2 is stabilisable by partial reset of order z .

Proof: By hypothesis, see Appendix 1 (specially Theorem A.3 and Corollary A.1), there exists an invertible matrix $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$, such that $T^{-1}A_p T = \begin{bmatrix} \tilde{A}_{p11} & \tilde{A}_{p12} \\ 0 & \tilde{A}_{p22} \end{bmatrix}$, $p \in \{1, 2\}$, where $\tilde{A}_{p11} \in \mathbb{R}^{(n-z) \times (n-z)}$ commute. Since T_{11} is invertible, by Theorem 3.2, then the switched system \mathbb{S} is stabilisable by partial reset of order z . ■

Next, we present an illustrative example that shows an application of the previous corollary, and motivates a slightly more general result as well as the algorithm that closes this section.

Example 4.1: Let \mathbb{S} be the switched system with switching bank associated to the stable matrices:

$$A_1 = \begin{bmatrix} -6.0 & 0 & 1.0 & 2.0 \\ -4.0 & -8.0 & -6.9 & 0.1 \\ 4.0 & 7.9 & 4.8 & -2.1 \\ -4.0 & -3.9 & -0.9 & 2.0 \end{bmatrix} \quad \text{and} \\ A_2 = \begin{bmatrix} -4.0 & -3.0 & -4.0 & -3.0 \\ -2.0 & -8.3109 & -8.5563 & -4.2454 \\ 2.0 & 8.2109 & 4.4563 & 0.24539 \\ -2.0 & -6.2109 & -2.5563 & -0.34539 \end{bmatrix}.$$

In the following, the presented results were obtained using floating-point computations in MATLAB (so, most of the equal signs must be interpreted as approximately equal). The matrix L associated to the matrices A_1 and A_2 , defined as $L = \sum_{k,l=1}^3 [A_1^k, A_2^l]^T [A_1^k, A_2^l]$, is a real 4×4 matrix such that $\text{rank } L = 3 < 4$, so A_1 and A_2 are partially commuting matrices. $\ker L$ is generated by $v_1 = [-0.5 \ -0.5 \ 0.5 \ -0.5]^T$. Since $-0.5 \neq 0$, we may conclude by the previous corollary that the system is stabilisable by partial reset of order 3. In fact, considering three more vectors, for instance, a set of generators of the orthogonal subspace of $\ker L$,

$$v_2 = [-0.5 \ 0.83333 \ 0.16667 \ -0.16667]^T \\ v_3 = [0.5 \ 0.16667 \ 0.83333 \ 0.16667]^T \\ v_4 = [-0.5 \ -0.16667 \ 0.16667 \ 0.83333]^T,$$

the matrix $T = [v_1 \ v_2 \ v_3 \ v_4]$ is such that

$$T^{-1}A_1 T = \begin{bmatrix} -5 & * \\ 0 & \tilde{B}_{122} \end{bmatrix} \quad \text{and} \quad T^{-1}A_2 T = \begin{bmatrix} -6 & * \\ 0 & \tilde{B}_{222} \end{bmatrix},$$

where $*$ denotes some matrices (with no special importance to us) and

$$\tilde{B}_{122} = \begin{bmatrix} -5.0333 & -4.9667 & -1.9333 \\ 4.9333 & 2.8667 & -0.066667 \\ -0.93333 & 1.0333 & -0.033333 \end{bmatrix} \quad \text{and} \\ \tilde{B}_{222} = \begin{bmatrix} -3.3555 & -3.6282 & -2.2727 \\ 3.2555 & -0.47183 & -1.7273 \\ -1.2555 & 2.3718 & 1.6273 \end{bmatrix};$$

Clearly, the blocks (1, 1): -5 and -6 have a CQLF.

Now, notice that the matrices \tilde{B}_{122} and \tilde{B}_{222} may be still partially commuting matrices (see Appendix 1 for details on this). This can be tested using exactly the procedure used above for the matrices A_1 and A_2 .

The matrix L associated to the matrices \tilde{B}_{122} and \tilde{B}_{222} , defined as $L = \sum_{k,l=1}^2 [\tilde{B}_{122}^k, \tilde{B}_{222}^l]^T [\tilde{B}_{122}^k, \tilde{B}_{222}^l]$, is such that $\text{rank } L = 2 < 3$ and its kernel is generated by $v_1 = [0.57735 \ -0.57735 \ 0.57735]^T$. Similarly, taking $V = \begin{bmatrix} 0.57735 & 0.57735 & -0.57735 \\ -0.57735 & 0.78868 & 0.21132 \\ 0.57735 & 0.21132 & 0.78868 \end{bmatrix}$, we update the matrix T as

$$T := T \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} -0.5 & -0.8660 & 0.0000 & 0.0000 \\ -0.5 & 0.2887 & 0.5774 & -0.5774 \\ 0.5 & -0.2887 & 0.7887 & 0.2113 \\ -0.5 & 0.2887 & 0.2113 & 0.7887 \end{bmatrix}.$$

Then, by computing $T^{-1}A_1 T$ and $T^{-1}A_2 T$, we arrive to a 2×2 upper block triangular structure:

$$T^{-1}A_1 T = \begin{bmatrix} -5.1 & -7.5056 & * \\ 0 & -2 & * \\ 0 & 0 & \tilde{B}_{133} \end{bmatrix} \quad \text{and} \\ T^{-1}A_2 T = \begin{bmatrix} -6.1 & -2.3094 & * \\ 0 & -2 & * \\ 0 & 0 & \tilde{B}_{233} \end{bmatrix},$$

where

$$\tilde{B}_{133} = \begin{bmatrix} -0.1 & -1.464 \\ 5.464 & -0.1 \end{bmatrix} \quad \text{and} \quad \tilde{B}_{233} = \begin{bmatrix} -1.832 & -2.464 \\ 4.464 & 1.632 \end{bmatrix}.$$

and the block (1, 1) of order 2 in T , $T_{11} = \begin{bmatrix} -0.5 & -0.8660 \\ -0.5 & 0.2887 \end{bmatrix}$ has clearly $\text{rank } 2 (=3 - 1)$. Furthermore, by construction (and as it can be easily seen), the 2×2 matrices in the left upper corners have CQLF. In this case we have obtained two triangular matrices. At this point we may conclude that the system is stabilisable by partial reset of order 2

and check if the remaining blocks \tilde{B}_{133} and \tilde{B}_{233} are still partially commuting matrices. The correspondent matrix $L = [\tilde{B}_{133}, \tilde{B}_{233}]^T [\tilde{B}_{133}, \tilde{B}_{233}]$ is such that $\text{rank } L = 2$. So, these blocks are no longer partially commuting. Using this iterative procedure we concluded that the system is stabilisable by partial reset of order 2.

As we have illustrated in the previous example, if two real matrices A_1 and A_2 are partially commuting matrices, they can be brought by a simultaneous similarity transformation, T , into block upper triangular matrices where a certain number of corresponding diagonal blocks commute, i.e.

$$T^{-1}A_1T = \begin{bmatrix} \tilde{A}_{111} & \tilde{A}_{112} \\ 0 & \tilde{A}_{122} \end{bmatrix} \text{ and} \\ T^{-1}A_2T = \begin{bmatrix} \tilde{A}_{211} & \tilde{A}_{212} \\ 0 & \tilde{A}_{222} \end{bmatrix}, \text{ respectively,}$$

where \tilde{A}_{111} and \tilde{A}_{211} are block triangular matrices where the corresponding diagonal blocks have the same order and commute (see Appendix 1 for details).

Definition 4.1: Let $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ be a switched system with switching bank associated to partially commuting stable matrices A_1 and A_2 . \mathbb{S} is called a $(n - z)$ -partially commuting system if there exists a real similarity transformation $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$, with $T_{11} \in \mathbb{R}^{(n-z) \times (n-z)}$ invertible, such that

$$T^{-1}A_pT = \begin{bmatrix} \tilde{A}_{p11} & \tilde{A}_{p12} \\ 0 & \tilde{A}_{p22} \end{bmatrix}, p = 1, 2,$$

where \tilde{A}_{111} and \tilde{A}_{211} are block triangular matrices, with the same block triangular structure, having commutative correspondent diagonal blocks.

Using this definition, by Corollary 2.1, Theorems 2.3 and 3.2, the following can be stated.

Theorem 4.2: Let $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ be a switched system associated to stable matrices A_p , $p = 1, 2$. If $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ is a $(n - z)$ -partially commuting switched system then \mathbb{S} is stabilisable by partial state reset of order z .

Based on the previous considerations, we present a constructive algorithm that tests if a given switched system is $(n - z)$ -partially commuting for some z , which implies the possibility of stabilisation by a partial reset of order z . More precisely, this algorithm returns the smallest order of a stabilising partial reset that can be constructed based on simultaneous block triangularisation with partially commuting diagonal blocks. The process is descending, until possible, and will have as many iterations as the number of blocks appearing in the

canonical forms presented in Theorem A.4 of Appendix A. In the algorithm, the matrices G_1 and G_2 are playing the same role as the blocks \tilde{B}_{1ii} and \tilde{B}_{2ii} , $i = 2, 3$, in Example 4.1, and the matrices A_1, A_2 are stable.

Algorithm

Initiation step: Set $T := I_n$, $G_1 := A_1$, $G_2 := A_2$, $n := (A_1)$, $g := n$ and $h := n$.

Step 1: Compute $L = \sum_{k,l=1}^{n-1} [G_1^k, G_2^l]^T [G_1^k, G_2^l]$ and set $r := \text{rank } L$.

If $r = 0$, write ‘the system is stable’ and stop.

If $r = g$, write ‘the system is stabilisable by partial reset of order h ’ and stop.

Else, go to Step 2.

Step 2: Compute a basis for $\ker L$ and denote it by $\{v_1, \dots, v_{g-r}\}$.

Step 3: Choose v_{g-r+1}, \dots, v_g such that $V := [v_1 \dots v_{g-r} \ v_{g-r+1} \dots v_g]$ is invertible. Set $V := \begin{bmatrix} I_{n-g} & 0 \\ 0 & V \end{bmatrix}$

Step 4: Set $T := TV$, $g := r$, $T_{11} := [I_{n-g} \ 0_{(n-g) \times g}] T \begin{bmatrix} I_{n-g} \\ 0_{g \times (n-g)} \end{bmatrix}$;

(a) If $\text{rank } T_{11} = n - g$, set $h := g$.

(b) Else, go to Step 5.

Step 5: Set $G_j := [0 \ I_g] T^{-1} A_j T \begin{bmatrix} 0 \\ I_g \end{bmatrix}$, $j = 1, 2$. Return to Step 1.

Remark 4.1: Note that this algorithm may fail to deliver the conclusion that the system is stabilisable by partial reset of order z even if the conditions of Theorem 3.2 are satisfied, as there exist simultaneously z -block triangularisable systems for which the blocks $(1, 1)$ of $T^{-1}A_pT$ have a CQLF, but are not partially commuting systems. In fact, $(n - z)$ -partially commuting systems with stable bank are a particular class of systems that satisfy the conditions of Theorem 3.2. On the other hand, in some cases, this procedure makes possible to conclude about the stability of the switched system, namely when it is easy to see that the final matrices G_1 and G_2 have a CQLF (cf. Theorem 2.3). This is, for instance, the case when G_1 and G_2 are 2×2 , and G_1G_2 and $G_1G_2^{-1}$ have no real negative eigenvalues (Shorten & Narendra, 2002), and the case when $\text{rank}(G_1 - G_2) = 1$ and G_1G_2 have no real negative eigenvalues (King & Nathanson, 2006). Note that, using these criteria, in Example 4.1, we may conclude the matrices A_1 and A_2 do not have CQLF, since the final two diagonal blocks do not have CQLF.

Remark 4.2: It is also worth noticing that this algorithm may be very sensitive to numerical precision, as it relies on rank computations. Since numerical perturbations are more likely to increase rather than decrease the rank, in most cases the algorithm leads to an overestimation of the order of the partial reset that achieves stabilisation. In

the worst-case scenario, the algorithm will deliver n as the order of such reset (that then became a total reset). However, taking into account that the algorithm only checks a sufficient condition we cannot conclude, without further research, that stabilisation by partial reset is itself a property which is sensitive to small numerical perturbations.

5. Conclusion

In this paper, we have considered switched systems that allow jumps in some state components when switching occurs. In particular, we have analysed the case where these jumps are dictated by a certain linear rule called a partial state reset. The central question was to verify if a switched system is stabilisable using partial reset of certain order. This question was examined for block simultaneously triangularisable switched systems. In particular, the stabilisation of partially commuting systems was analysed. An algorithm was proposed that determines the smallest order of a stabilising partial reset that can be constructed based on simultaneous block triangularisation with partially commuting diagonal blocks. This criterion is somehow a generalisation of the sufficient condition presented in Narendra and Balakrishnan (1994), for stability of switched systems involving commutative matrices. We note that the obtained conditions based on partial commutation or on simultaneous block triangularisation are still restrictive. Nevertheless, we suspect that obtaining general conditions, within an algebraic approach (like the one used here) will prove to be difficult.

Notes

1. The situation where $R_{22}^{(q,p)}$ are not invertible raises interesting questions that will be investigated in due time.
2. $[X, Y]$ stands for the traditional commutator of two square matrices X and Y , i.e. $[X, Y] = XY - YX$.

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Appendix A. Maximal invariant subspace where a finite set of matrices commute and partially commuting matrices

Let $\mathcal{A} = \{A_p : p \in \mathcal{P}\}$ be a set of real matrices of order n and \mathcal{M} be a subspace of $\mathbb{R}^{n \times n}$. \mathcal{M} is said to be \mathcal{A} -invariant if it is A_p -invariant for ever $p \in \mathcal{P}$. The matrices A_p , $p \in \mathcal{P}$ are said to commute (pairwise) on \mathcal{M} if $(A_i A_j - A_j A_i)x = 0$, for all $x \in \mathcal{M}$. Note that there exists always an invariant subspace where a set of matrices commute, that subspace is the zero space. The existence and characterisation of non-trivial subspaces with such properties were first studied in Shemesh (1984), for sets of two complex matrices. In fact, in that paper, Shemesh was mainly concerned with an equivalent problem: the existence of a common eigenvector of two complex matrices. The characterisation of the sets of two matrices with a common eigenvector is the following, where $[X, Y]$ is the standard notation for the commutator $XY - YX$ of the matrices X and Y :

Theorem A.1 (Shemesh (1984)): *The matrices $A_1, A_2 \in \mathbb{C}^{n \times n}$ have a common eigenvector if and only if*

$$\mathcal{N} = \bigcap_{i,j=1}^{n-1} \ker [A_1^i, A_2^j] \neq \{0\}.$$

Recently, the following generalisation of Shemesh's theorem for a higher number of matrices has been given, its proof essentially mimics the Shemesh's proof.

Theorem A.2 (Jamiołkowski & Pastuszak, 2015): *Let A_1, A_2, \dots, A_N be a set matrices in $\mathbb{C}^{n \times n}$. The matrices A_1, A_2, \dots, A_N have a common eigenvector if and only if*

$$\mathcal{N} = \bigcap_{\substack{k_1 + k_2 + \dots + k_s \neq 0 \\ l_1 + l_2 + \dots + l_s \neq 0 \\ k_j, l_j \geq 0}}^{n-1} \ker [A_1^{k_1} \dots A_s^{k_s}, A_1^{l_1} \dots A_s^{l_s}] \neq \{0\}.$$

For our purposes, the properties of the non-zero subspace \mathcal{N} are more important that the theorems themselves. Furthermore, since our matrices are real and we want to work with subspaces in \mathbb{R}^n , we must carefully analyse the arguments used in order to see if they still hold. For sake of simplicity, we consider the case $N = 2$. The next result reflects that analysis; the proof that we present is slightly different from the one presented in Shemesh (1984).

Theorem A.3: *Let A_1, A_2 be matrices in $\mathbb{R}^{n \times n}$ such that $\mathcal{N} = \bigcap_{i,j=1}^{n-1} \ker [A_1^i, A_2^j]$ is a non-zero subspace of \mathbb{R}^n . Then \mathcal{N} is an $\{A_1, A_2\}$ -invariant subspace where A_1 and A_2 commute. Moreover, \mathcal{N} is the maximal subspace with that property.*

Proof: First note that, using Cayley–Hamilton theorem, $\mathcal{N} = \bigcap_{i,j \in \mathbb{N}} \ker [A_1^i, A_2^j]$. If $x \in \mathcal{N}$,

$$\begin{aligned} [A_1^i, A_2^j]A_1x &= A_1^i A_2^j A_1x - A_2^j A_1^i A_1x \\ &= A_1^i A_1 A_2^j x - A_2^j A_1^{i+1} x \\ &= 0, \text{ for every } i, j \in \mathbb{N}. \end{aligned}$$

So, \mathcal{N} is A_1 -invariant. Similarly, it can be shown that \mathcal{N} is A_2 -invariant. Therefore, \mathcal{N} is $\{A_1, A_2\}$ -invariant. The commutativity of the matrices on \mathcal{N} is trivial, since if $x \in \mathcal{N}$, $x \in \ker [A_1, A_2]$.

It remains to prove that \mathcal{N} is the maximal invariant subspace where A_1 and A_2 commute. Let W be a subspace of \mathbb{R}^n such that W is $\{A_1, A_2\}$ -invariant and $A_1 A_2 x = A_2 A_1 x$, for every $x \in W$. Next we prove that $W \subset \mathcal{N}$ by proving that $W \subset \ker [A_1^i, A_2^j]$, for every $i, j \in \mathbb{N}$. First, let us prove that, by induction,

$$W \subset \ker [A_1^i, A_2], i \in \mathbb{N}. \quad (\text{A1})$$

In fact, for $i = 1$ (A1) holds. Suppose that $W \subset \ker[A_1^k, A_2]$, i.e. for all $x \in W$, $A_1^k A_2 x = A_2 A_1^k x$. Then, for all $x \in W$,

$$\begin{aligned} A_1^{k+1} A_2 x - A_2 A_1^{k+1} x &= A_1^{k+1} A_2 x - A_2 A_1^k \underbrace{A_1 x}_{\in W} \\ &= A_1^{k+1} A_2 x - A_1^k A_2 A_1 x \\ &= A_1^{k+1} A_2 x - A_1^{k+1} A_2 x \\ &= 0 \end{aligned}$$

So $W \subset \ker[A_1^{k+1}, A_2]$. Therefore, by induction, that $W \subset \ker[A_1^i, A_2]$, $i \in \mathbb{N}$. Now, let us consider a fixed $i \in \mathbb{N}$, and prove, by induction (considering that (A1) holds), that

$$W \subset \ker[A_1^i, A_2^j], j \in \mathbb{N}. \quad (\text{A2})$$

In fact, for $j = 1$ (A2) holds, because (A1) holds. Suppose that $W \subset \ker[A_1^i, A_2^k]$, i.e. for all $x \in W$, $A_1^i A_2^k x = A_2^k A_1^i x$. Then, for all $x \in W$,

$$\begin{aligned} A_1^i A_2^{k+1} x - A_2^{k+1} A_1^i x &= A_1^i A_2^k \underbrace{A_2 x}_{\in W} - A_2^{k+1} A_1^i x \\ &= A_2^k A_1^i A_2 x - A_2^{k+1} A_1^i x \\ &= A_2^{k+1} A_1^i x - A_2^{k+1} A_1^i x \\ &= 0 \end{aligned}$$

So, $W \subset \ker[A_1^i, A_2^{k+1}]$. Therefore, by induction, it follows that $W \subset \ker[A_1^i, A_2^j]$, $i, j \in \mathbb{N}$. ■

Notice that the subspace \mathcal{N} may be written as $\mathcal{N} = \ker L$, where

$$L = \sum_{i,j=1}^{n-1} [A_1^i, A_2^j]^T [A_1^i, A_2^j]$$

and $\dim \mathcal{N} = n - r$, with $r = \text{rank } L$. Consequently, the next corollary follows:

Corollary A.1: Let $\mathcal{A} = \{A_1, A_2\}$ be a set matrices in $\mathbb{R}^{n \times n}$. If $\text{rank } L = r$, for $r \neq n$, then there exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that

$$T^{-1} A_1 T = \begin{bmatrix} B_{111} & \tilde{B}_{112} \\ 0 & \tilde{B}_{122} \end{bmatrix} \quad (\text{A3})$$

$$T^{-1} A_2 T = \begin{bmatrix} B_{211} & \tilde{B}_{212} \\ 0 & \tilde{B}_{222} \end{bmatrix}, \quad (\text{A4})$$

where the blocks $B_{111}, B_{211} \in \mathbb{R}^{(n-r) \times (n-r)}$ commute.

This way of obtaining block triangular matrices by a simultaneous similarity transformation, where the blocks (1, 1) commute, is described in Alpin et al. (2000). In this paper, matrices that have a common eigenvector (which is equivalent to $\mathcal{N} \neq \{0\}$) are called *partially commutative matrices*. Notice that, if $r = 0$ the initial matrices are already commutative, and in this degenerated case the blocks (2, 2) must be considered missing and T may be considered to be the identity matrix. Otherwise, the blocks \tilde{B}_{122} and \tilde{B}_{222} may be commutative or not. If they are not commutative, they might be partially commutative matrices. In case \tilde{B}_{122} and \tilde{B}_{222} are partially commutative matrices, the same type of simultaneous block triangularisation is possible. Therefore, in that case, there exists an invertible matrix S such that

$$S^{-1} A_1 S = \begin{bmatrix} B_{111} & B_{112} & \tilde{B}_{113} \\ 0 & B_{122} & \tilde{B}_{123} \\ 0 & 0 & \tilde{B}_{133} \end{bmatrix} \quad (\text{A5})$$

$$S^{-1} A_2 S = \begin{bmatrix} B_{211} & B_{212} & \tilde{B}_{213} \\ 0 & B_{222} & \tilde{B}_{223} \\ 0 & 0 & \tilde{B}_{233} \end{bmatrix}, \quad (\text{A6})$$

where the blocks B_{1ii}, B_{2ii} , $i = 1, 2$, commute. This process may be repeated (depending on partial commutativity of the blocks (3, 3)) and so on until possible. Eventually, with this procedure, A_1 and A_2 will be brought by a simultaneous similarity transformations into block upper triangular matrices where a certain number of corresponding diagonal blocks commute. In order to be more precise, we state the following theorem, which is the version for real matrices of the one stated for complex matrices in (Alpin et al., 2000).

Theorem A.4: Let $A_1, A_2 \in \mathbb{R}^{n \times n}$ be partially commutative matrices. Then, for some positive integer k , there exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that

$$T^{-1} A_1 T = \begin{bmatrix} B_{111} & B_{112} & \dots & B_{11k} \\ 0 & B_{122} & \dots & B_{12k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{1kk} \end{bmatrix} \quad (\text{A7})$$

$$T^{-1} A_2 T = \begin{bmatrix} B_{211} & B_{212} & \dots & B_{21k} \\ 0 & B_{222} & \dots & B_{22k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{2kk} \end{bmatrix}, \quad (\text{A8})$$

where the blocks B_{1ii}, B_{2ii} , $i = 1, \dots, k-1$, commute. The last blocks B_{1kk} and B_{2kk} either commute or do not partially commute. The number of blocks and its sizes are uniquely determined by A_1 and A_2 .